Ladder Operators for Integrable One-Dimensional Lattice Models

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Abstract

A generalised ladder operator is used to construct the conserved operators for any one-dimensional lattice model derived from the Yang-Baxter equation. As an example, the low order conserved operators for the XYh model are calculated explicitly.

1 Introduction

The method for constructing integrable one-dimensional lattice models from solutions of the Yang-Baxter equation is well known (eg. see [1]). In principle, the conserved operators can be obtained by series expansion of the family of commuting transfer matrices. A more practical approach is to use the ladder operator which permits a recursive method through repeated commutators to obtain the conserved operators.

For models where the solution of the Yang-Baxter equation has the difference property, it has been established [2, 3, 4] that the ladder operator is a lattice analogue of the boost operator for Lorentz invariant systems. Recently it has been shown that for the Hubbard model, which is not Lorentz invariant in the continuum limit as a consequence of spin-charge separation, and is reflected by the fact that the solution of the Yang-Baxter equation does not have the difference property, the ladder operator still exists [5].

The present work extends [5] to develop a general theory for the construction of the ladder operator for any integrable system obtained through the Yang-Baxter equation. The theory will be applied to analyse the conservation laws for the XY model in a transverse magnetic field.

2 Integrable Lattice Models using the Quantum Inverse Scattering Method

We begin with a vector-dependent solution of the Yang-Baxter equation

$$R_{12}(\vec{u},\vec{v})R_{13}(\vec{u},\vec{w})R_{23}(\vec{v},\vec{w}) = R_{23}(\vec{v},\vec{w})R_{13}(\vec{u},\vec{w})R_{12}(\vec{u},\vec{v})$$

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where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are m-component vectors. Throughout, we assume the regularity property $R(\vec{u},\vec{u}) = P$. Define a set of m local Hamiltonians

$$h_l\{i\} = P. \left.\frac{\partial R_{l(l+1)}(\vec{u},\vec{v})}{\partial u_i}\right|_{\vec{u}=\vec{v}}, \quad i = 1,..,m$$

with the corresponding global Hamiltonians acting on a one-dimensional lattice of length L given by

$$H\{i\} = \sum_{l=0}^{L-1} h_l\{i\}.$$

Throughout, periodic boundary conditions are assumed on all summations which are evaluated over the length of the lattice. Note it is implicit that all the operators $h\{i\}$ are in fact functions of $\vec{v}$.

The transfer matrix is constructed through

$$T(\vec{u},\vec{v}) = \mathrm{tr}_a \left(R_{a(L-1)}(\vec{u},\vec{v})...R_{a1}(\vec{u},\vec{v})R_{a0}(\vec{u},\vec{v})\right)$$

where a refers to the auxiliary space, which by the standard argument gives rise to a commutative family in the first variable; i.e.

$$[T(\vec{u},\vec{v}),\, T(\vec{w},\, \vec{v})] = 0, \quad \forall\, \vec{u},\vec{w}. \tag{1}$$

It can also be easily verified that

$$[H\{i\},\, T(\vec{u},\vec{v})] = 0, \quad \forall\, \vec{u}. \tag{2}$$

It is convenient, however, to define the conserved operators as

$$t\{\vec{n}\} = \left[\frac{\partial^{n_1+...+n_m}}{\partial u_1^{n_1}...\partial u_m^{n_m}} \ln T(\vec{u},\vec{v})\right]$$

where they appear in the series expansion

$$\ln T(\vec{u},\vec{v}) = \sum_{\vec{n}} \frac{(u_1-v_1)^{n_1}...(u_m-v_m)^{n_m}}{n_1!...n_m!}\, t\{\vec{n}\}. \tag{3}$$

Thus it follows from (1) that

$$[t\{\vec{n}\},\, t\{\vec{k}\}] = 0, \quad \forall\, \vec{n},\vec{k}$$

and moreover from (2)

$$[H\{i\},\, t\{\vec{n}\}] = 0, \quad \forall\, i,\vec{n}.$$

Note that $\vec{n}$ is an m-component vector with non-negative integer entries. Introducing the notation $\{\vec{\epsilon}_i\}_{i=1}^m$ for the basis of the m-dimensional vector space, we can write

$$\vec{n} = \sum_{i=1}^m n_i\vec{\epsilon}_i.$$

3 Recursion Formula for Calculating the Conserved Operators

For each of the index labels i we define a ladder operator

$$B\{i\} = \sum_{l=0}^{L-1} l h_l\{i\}$$

with the coefficients l taken from the set of integers modulo L. For any function ϕ admitting a Taylor's series expansion we have

$$[B\{i\},\, \phi(\mathcal{T})] = \mathcal{T}.H\{i\}.\phi'(\mathcal{T})$$

where $\mathcal{T} = T(\vec{u}, \vec{u})$ and ϕ' denotes the derivative of ϕ. Choosing ϕ to be the logarithm now gives

$$[B\{i\},\, \ln \mathcal{T}] = H\{i\}.$$

It can be shown that

$$[B\{i\},\, T(\vec{u}, \vec{v})] = -\frac{\partial T(\vec{u}, \vec{v})}{\partial v_i}. \tag{4}$$

As a result we obtain the following recursion formula from (4) and the expansion (3)

$$t\{\vec{n} + \vec{\epsilon}_i\} = [B\{i\},\, t\{\vec{n}\}] + \frac{\partial t\{\vec{n}\}}{\partial v_i} \tag{5}$$

The first few terms in (3) can be identified immediately

$$t\{\vec{0}\} = \ln \mathcal{T}, \qquad t\{\vec{\epsilon}_i\} = H\{i\}. \tag{6}$$

In principle, through repeated use of (4) expressions for all the operators $t\{\vec{n}\}$ may be obtained.

Applying the recursion (5), the second order conserved currents can be obtained by the following formula:

$$\begin{aligned} t\{\vec{\epsilon}_i + \vec{\epsilon}_j\} &= \frac{1}{2}\sum_l [h_l\{j\},\, h_{l-1}\{i\}] + \frac{1}{2}\sum_l [h_l\{i\},\, h_{l-1}\{j\}] \\ &\quad + \frac{1}{2}\frac{\partial H\{j\}}{\partial v_i} + \frac{1}{2}\frac{\partial H\{i\}}{\partial v_j}. \end{aligned} \tag{7}$$

4 The XYh Model

The XY model in a transverse magnetic field has the following Hamiltonian:

$$H = \sum_{i=1}^{N} (J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y + h\sigma_i^z) \qquad J_x, J_y, h \text{ const.}$$

This model is known to be integrable [6]. Barouch and Fuchssteiner [7], Araki [8], and Grabowski and Mathieu [9] have explicitly calculated the low order conserved operators. These results have been reproduced using the generalised ladder operator method.

4.1 R Matrix of the XYh Model

Bazhanov and Stroganov [10] constructed an elliptic parametrization for the Boltzmann vertex weights of the XYh model. In this parametrization, the weights are meromorphic functions of 3 complex variables, $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$, where only the first vector entry contains the difference property.

The R matrix is

$$R(\vec{u}, \vec{v}) = \begin{pmatrix} R_{11}^{11} & 0 & 0 & R_{22}^{11} \\ 0 & R_{12}^{12} & R_{21}^{12} & 0 \\ 0 & R_{12}^{21} & R_{21}^{21} & 0 \\ R_{11}^{22} & 0 & 0 & R_{22}^{22} \end{pmatrix}$$

with

$$\begin{aligned} R_{22}^{22} &= \rho(1 - e(u_1 - v_1)e(u_2)e(v_2)) \qquad R_{11}^{11} = \rho(e(u_1 - v_1) - e(u_2)e(v_2)) \\ R_{21}^{21} &= \rho(e(u_2) - e(u_1 - v_1)e(v_2)) \qquad R_{12}^{12} = \rho(e(v_2) - e(u_1 - v_1)e(u_2)) \\ R_{21}^{12} &= R_{12}^{21} = \frac{\rho\sqrt{e(u_2)s(u_2)}\sqrt{e(v_2)s(v_2)}(1 - e(u_1 - v_1))}{s\left(\frac{u_1 - v_1}{2}\right)} \\ R_{22}^{11} &= R_{11}^{22} = -ik\rho\sqrt{e(u_2)s(u_2)}\sqrt{e(v_2)s(v_2)}(1 + e(u_1 - v_1))s\left(\frac{u_1 - v_1}{2}\right) \end{aligned}$$

where ρ is an arbitrary constant and s and e are the respective elliptic functions sn and $(cn + i\, sn)$. By imposing the condition $R(\vec{u}, \vec{u}) = P$, we obtain the value of $\rho = \frac{1}{1 - e^2(v_2)}$.

4.2 Local Hamiltonians

The local Hamiltonians $h_l\{1\}$ and $h_l\{2\}$ are given as follows:

$$\begin{aligned} h_l\{1\} &= \rho[A(\sigma^x \otimes \sigma^x) + B(\sigma^y \otimes \sigma^y) + C(I \otimes I) + D(I \otimes \sigma^z + \sigma^z \otimes I)] \\ h_l\{2\} &= \rho[E(I \otimes I) + F(\sigma^x \otimes \sigma^y - \sigma^y \otimes \sigma^x)] \end{aligned}$$

where $\sigma^x, \sigma^y, \sigma^z$ are the Pauli sigma matrices and

$$\begin{aligned}
A &= -\frac{1}{2}ie(v_2)(1+ks(v_2)), & B &= -\frac{1}{2}ie(v_2)(1-ks(v_2)), & k \text{ const.}\\
C &= s(v_2)e(v_2), & D &= \frac{1}{2}ic(v_2)e(v_2),\\
E &= -id(v_2)e(v_2)^2, & F &= \frac{1}{2}d(v_2)e(v_2).
\end{aligned}$$

where d and c are the elliptic functions dn and cn respectively.

4.3 Second Order Conserved Currents

The second order conserved currents can be obtained by the formula (7):

$$\begin{aligned}
t\{2\vec{\epsilon}_1\} = t\{2\vec{\epsilon}_2\} &= \sum_l \{\alpha(\sigma^x\otimes\sigma^z\otimes\sigma^y - \sigma^y\otimes\sigma^z\otimes\sigma^x)\\
&\quad + \beta(\sigma^x\otimes\sigma^y\otimes I - \sigma^y\otimes\sigma^x\otimes I)\} \qquad \alpha,\beta \text{ const.}
\end{aligned}$$

and

$$\begin{aligned}
t\{\vec{\epsilon}_1+\vec{\epsilon}_2\} &= \sum_l \{\gamma(\sigma^x\otimes\sigma^z\otimes\sigma^x) + \zeta(\sigma^y\otimes\sigma^z\otimes\sigma^y)\\
&\quad + \eta(\sigma^x\otimes\sigma^x\otimes I + \sigma^y\otimes\sigma^y\otimes I)\\
&\quad - (\gamma+\zeta)(\sigma^z\otimes I\otimes I)\} \qquad \gamma,\zeta,\eta \text{ const.}
\end{aligned}$$

in agreement with [7]-[9].

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